National Examination – Dec 2015
04-BS-16: Discrete Mathematics
Duration: 3 hours

Examination Type: Closed Book.
No aids allowed.

This exam paper contains 13 pages (including this one).
Answer 10 out of 12 questions. Ten questions constitute a full paper.
Please clearly indicate which two questions you don't want marked by drawing a diagonal line across the page.
In case of doubt to any question, clearly state any assumptions made.
One of two calculators is permitted any Casio or Sharp approved models.

# 1:	______/ 10
# 2:	______/ 10
# 3:	______/ 10
# 4:	______/ 10
# 5:	______/ 10
# 6:	______/ 10
# 7:	______/ 10
# 8:	______/ 10
# 9:	______/ 10
# 10:	______/ 10
# 11:	______/ 10
# 12:	______/ 10
TOTAL:	______/100

Good Luck!

Question 1. [10 MARKS]

Part (a) [2 MARKS]

Rewrite the following without negation on qualifiers $\neg\exists x\neg\forall y\neg\exists z P(x,y,z)$

Part (b) [2 MARKS]

Write the sentence "A necessary condition for $P(x,y)$ to be true is that $x > y$" as a logic expression.

Part (c) [3 MARKS]

Is $\exists x\forall y P(x,y) \rightarrow \forall x\exists y P(x,y)$ a tautology? Please either provide a proof or give a counterexample.

Part (d) [3 MARKS]

Consider the universe of discourse as positive intergers. Let $P_n(x,y,z)$ stand for $x^n + y^n = z^n$. Write the Fermat's Last Theorem as a logical proposition, i.e., the equation $x^n + y^n = z^n$ does not have positive integer solution for $n > 2$.

Question 2. [10 MARKS]

Part (a) [5 MARKS]

Show that

$$\sum_{s_1=0}^{1}\sum_{s_2=0}^{1}\cdots\sum_{s_n=0}^{1}\frac{1}{1^{s_1}2^{s_2}\cdots n^{s_n}} = n+1$$

Part (b) [5 MARKS]

Show that the sum of even numbers from 0, 2, $\cdots$ to $2n$ is $n(n+1)$.

Question 3. [10 MARKS]

Consider a sequence recursively defined as follows: $a_0 = 2$, $a_{n+1} = a_n^2$.

Part (a) [2 MARKS]

Write down a closed-form expression for a_n.

Part (b) [3 MARKS]

Is $a_n = O(2^n)$? Is $a_n = O(n^n)$?

Part (c) [5 MARKS]

Prove that $a_n - 1$ has at least n distinct prime divisors.

Question 4. [10 MARKS]

A 5-card poker hand is dealt from a 52-card deck. Find the probability of getting

a. Five cards of consecutive rank (2 is the smallest rank, A largest).

b. There is at least one card of each suite.

c. All five cards come from the same suite.

d. There is exactly one pair.

e. Full house: three cards of same rank, plus a pair of different rank.

Question 5. [10 MARKS]

In the world series, two teams play a sequence of up to 7 games. The first team that wins 4 games wins the series. Assume that the teams are evenly matched.

Part (a) [2 MARKS]

What is the probability that the series ends after 4 games?

Part (b) [3 MARKS]

What is the probability that the series ends after the 5th game?

Part (c) [3 MARKS]

What is the probability that the series ends after the 6th game?

Part (d) [2 MARKS]

What is the probability that the series goes to the 7th game?

Question 6. [10 MARKS]

Part (a) [6 MARKS]

Suppose that we have 6 men and 4 women. How many different ways that

a. They can sit in a circular table so that all women sit next to each other? (clockwise and counter-clockwise seatings are regarded as different)

b. A committee of 5 people can be formed so that at most one of John, Mary and Susan is on the committee?

c. A committee of 5 people can be formed with more women than men?

Part (b) [4 MARKS]

How many ways there are to re-arrange the letters in SCIENCE, if

a. there are no restrictions?

b. the C's are together

Question 7. [10 MARKS]

Part (a) [4 MARKS]

Consider the relation R defined on real numbers, where $(a, b) \in R$ if and only if $a - b$ is an integer. Show that R is an equivalence relation. Describe the equivalence classes.

Part (b) [6 MARKS]

Plot the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \sin(x) + x$ over $x \in [-10, 10]$. Is this function one-to-one? onto? Does it have an inverse? If not, specify the largest sets $\mathcal{X}$ and $\mathcal{Y}$ for which the function $f : \mathcal{X} \to \mathcal{Y}$ has an inverse.

Question 8. [10 MARKS]

Part (a) [6 MARKS]

Show that a Fibonacci sequence with the initial condition $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ can be written in closed-form as

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Part (b) [4 MARKS]

Prove that $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Question 9. [10 MARKS]

Part (a) [2 MARKS]

Provide a definition of what it means by $f(n)$ is $O(g(n))$?

Part (b) [4 MARKS]

Insertion sort builds a sorted list by inserting one item to the list at a time. Describe how the algorithm works. What is the best-case, the worst-case, and the average run-time complexity of insertion sort? Please explain and provide adequate justification.

Part (c) [1 MARK]

Write down the name of a sorting algorithm that has better average run-time complexity than insertion sort.

Part (d) [3 MARKS]

Please order the following run-time complexity in big-O notation from slowest to fastest.

$$O(n^2),\ O(n\sqrt{n}),\ O(\log(n)),\ O((\log(n))^2),\ O(\log(\log(n))),\ O(2^n),\ O(n^2\log(n)),\ O(1)$$

Question 10. [10 MARKS]

Part (a) [2 MARKS]

Let G be a connected planar simple graph with e edges, and v vertices. Let f be the number of regions in the planar representation of G (including the outer region). What is the relation between e, f and v?

Part (b) [2 MARKS]

A truncated tetrahedron has 4 hexagonal faces and 4 triangle faces. How many vertices and how many edges does it have?

Part (c) [3 MARKS]

Suppose that you use 20 equilateral triangles of same size as faces to construct a polyhedron, you will get a regular icosahedron. How many triangles meet around each vertex?

Part (d) [3 MARKS]

A truncated rhombic dodecahedron consists of square faces and hexagon faces. It has 48 edges and 32 vertices. How many faces are squares and how many hexagons?

Question 11. [10 MARKS]

Part (a) [2 MARKS]

What is an Euler circuit of a graph? Under what condition does a graph have a Euler circuit?

Part (b) [3 MARKS]

For what values of (m, n) does $K_{m,n}$, the complete bipartite graph with m vertices on one side and n vertices on the other, have a Euler circuit? Explain.

Part (c) [2 MARKS]

What is a Hamilton path of a graph?

Part (d) [3 MARKS]

Illustrate whether tetrahedron (four triangle faces), cube (six square faces), and octahedron (eight triangle faces) have a Hamilton path or not.

Question 12. [10 MARKS]

In some cultures, families prefer boys to girls. Suppose that in a society all families keep having more children until a boy is born (and they stop having children as soon as a boy is born). Assume that boys and girls are born with equal probability.

Part (a) [3 MARKS]

Give an expression for the average number of children per family in this society.

Part (b) [2 MARKS]

Give an expression for the average number of girls per family in this society.

Part (c) [1 MARK]

What is the average number of boys per family in this society?

Part (d) [3 MARKS]

Would this society have an imbalance between males and females in the population over the long run? Please explain why or why not.

Total Marks = 100