National Examination – Dec 2013
04-BS-16: Discrete Mathematics
Duration: 3 hours

Examination Type: Closed Book.
No aids allowed.

Last Name: ______________________________

First Name: ______________________________

*Do **not** turn this page until you have received the signal to start.*
(In the meantime, please fill out the identification section above, and read the instructions below.)

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# 10:	______/ 10
# 11:	______/ 10
# 12:	______/ 10
TOTAL:	______/100

This exam paper contains 13 pages (including this one).
Answer 10 out of 12 questions. Ten questions constitute a full paper.
Please clearly indicate which two questions you don't want marked by drawing a diagonal line across the page.
In case of doubt to any question, clearly state any assumptions made.

Good Luck!

Question 1. [10 MARKS]

Let the universe of discourse be set of integers. Define the following propositions as:

- $P(x)$: "x is prime"
- $L(x, y)$: "x is less than y"
- $E(x)$: "x is even"
- $M(x, y, z)$: "$x \cdot y = z$"

Part (a) [6 MARKS]

Express each of the statements below in simple English:

a. $\exists x(P(x) \wedge E(x))$

b. $\forall x \forall y(P(x) \wedge P(y) \wedge L(x, y) \rightarrow \neg E(y))$

c. $\forall x \exists y(L(x, y) \wedge P(y))$

Part (b) [4 MARKS]

Write the following statement using predicates, quantifiers and logic connectives.

a. The product of two primes is not prime.

b. Every positive integer has a prime factor

Question 2. [10 MARKS]

Let $A = \{1, 2, 3\}$, $B = \{1, \{2\}\}$ and $C = \{\emptyset\}$.

Part (a) [6 MARKS]

Write down

- $A \cup B$
- $A \cap B$
- $A - B$
- $B \times C$
- $|C|$
- Power set of B

Part (b) [4 MARKS]

Determine whether the following statements are True or False:

- $2 \in B$
- $\emptyset \in C$
- $\emptyset \subseteq C$
- $|A| = |B \cup C|$

Question 3. [10 MARKS]

Part (a) [5 MARKS]

The harmonic numbers are defined as $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Use mathematical induction to prove that

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n$$

Part (b) [5 MARKS]

If 10 integers are chosen from the set $\{1, 2, \cdots, 99\}$, prove that there are at least two, say x and y, such tha $|\sqrt{x} - \sqrt{y}| < 1$.

Question 4. [10 MARKS]

Consider the relation R where $(a, b) \in R$ iff $a|b$ (i.e., a divides b).

Part (a) [2 MARKS]

Let R be defined on the set $\{1, 2, 3, 6, 12\}$. Draw a diagram representing R using arrows to connect a to b whenever $(a, b) \in R$

Part (b) [8 MARKS]

Now consider R defined on positive integers.

a. Is R reflexive? Justify.

b. Is R symmetric? Justify.

c. Is R transitive? Justify.

d. Is R an equivalence relation? Justify.

Question 5. [10 MARKS]

Define $f : \mathbb{R} - \mathbb{Z} \to \mathbb{R}$ as $f(x) = 4x + 2$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers.

a. Is f a function? Explain.

b. Give the domain and range of f.

c. Is f one-to-one? Justify your answer.

d. Is f onto? Justify your answer.

e. Does f have an inverse? If not, specify for what set Y, the function $f : \mathbb{R} - \mathbb{Z} \to Y$, $f(x) = 4x + 2$, has an inverse, and give the inverse.

Question 6. [10 MARKS]

Part (a) [7 MARKS]

In how many ways can all the letters of the word ONTARIO be arranged if

a. there are no restrictions?

b. the O's must be together?

c. all the vowels must be together?

Part (b) [3 MARKS]

Given the equation $x_1 + x_2 + x_3 + x_4 = 15$, find the number of integer solutions if

a. all the variables must be non-negative

b. all the variables must be greater than one

Question 7. [10 MARKS]

Consider the experiment of rolling two fair dices

Part (a) [6 MARKS]

a. What is the probability that exactly one die shows six?

b. What is the probability that the sum of two dices is greater than three?

c. What is the probability that the total is less than 7 given that at least one die shows a two

Part (b) [4 MARKS]

Assuming that in a four-child family, all sixteen possible sequence of boys/girls are equally likely,

a. Find the probability that there are four girls given that the youngest child is a girl.

b. Find the probability that there are four girls given that there is at least one girl.

Question 8. [10 MARKS]

Part (a) [4 MARKS]

Find an expression in terms of x, $x \in \mathbb{R}, x \geq 0$, for *exactly* how many times the square root operation sqrt() is executed in the following algorithm.

```
y=x
while(y>=2){
  y = sqrt(y)
}
```

Part (b) [2 MARKS]

Give a formal definition of $f(n) = O(g(n))$.

Part (c) [4 MARKS]

Order the following functions from the slowest to fastest growing: $\sqrt{n}$, $\log(n)$, n^3, $\log(\log(n))$, n^n, $n\log(n)$, 2^n, $(\log(n))^2$.

Question 9. [10 MARKS]

Part (a) [4 MARKS]

An Euler path is a path in a graph which visits every edge exactly once.

a. Does $K_{2,3}$, the complete bipartite graph with 2 vertices on one side and 3 vertices on the other, have a Euler path? Explain.

b. Under what condition does a graph have a Euler path?

Part (b) [6 MARKS]

For each of the following graphs, explain whether they are planar or not.

a. $K_{3,3}$, the complete bipartite graph with 3 vertices on one side and 3 vertices on the other.

b. K_4, the complete graph with 4 vertices.

c. K_5, the complete graph with 5 vertices.

Question 10. [10 MARKS]

Part (a) [3 MARKS]

Explain how insertion sort works.

Part (b) [3 MARKS]

Suppose that a list of size n is already sorted. How many comparison operations are executed if we apply insertion sort to this list?

Part (c) [2 MARKS]

Suppose that a list of size n is reversely sorted. How many comparison operations are executed if we apply insertion sort to this list?

Part (d) [2 MARKS]

What is the worst case computational complexity of insertion sort in big-O notation as a function of the size of the list?

Question 11. [10 MARKS]

Part (a) [3 MARKS]

A finite, connected, planar graph is drawn in the plane without any edge intersections. Let v be the number of vertices, e be the number of edges, and f be the number of faces (i.e., regions bounded by edges, including the outer, infinitely large region). Write down a mathematical relationship between v, e and f.

Part (b) [3 MARKS]

A truncated tetrahedron has 4 regular hexagonal faces, and 4 equilateral triangle faces. How many edges and how many vertices does it have?

Part (c) [4 MARKS]

A truncated octahedron is a polyhedron with 24 vertices and 36 edges constructed with regular hexagonal faces and square faces. How many hexagonal and how many square faces are in the truncated octahedron?

Question 12. [10 MARKS]

Part (a) [4 MARKS]

a. Determine the chromatic number of K_n, complete graph of n vertices.

b. Determine the chromatic number of $K_{m,n}$, complete bipartite graph with n vertices on one side and m vertices on the other.

Part (b) [6 MARKS]

Suppose that we need to schedule final exams for 7 courses numbered 1 to 7. The following pairs of courses have students in common: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,7\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{2,7\}$, $\{3,4\}$, $\{3,6\}$, $\{3,7\}$, $\{4,5\}$, $\{4,6\}$, $\{5,6\}$, $\{5,7\}$, $\{6,7\}$. What is the minimum number of time slots that are needed so that no student has two exams at the same time? Show which courses should be scheduled in the same time slots in the optimal solution.

Total Marks = 100