National Exams December 2010

98-Phys-A2, Statistical Physics

3 hours duration

Notes:

1. If doubt exists as the the interpretation of any question, the candidate is urged to submit with the answer paper, a clear statement of any assumptions made.
2. Candidates may use one of two calculators, the Casio or Sharp approved models. This is a Closed Book exam. One aid sheet written on both sides is permitted.
3. Any 4 (FOUR) questions constitute a complete paper. If more than 4 questions are attempted, indicate which questions are to be marked. Otherwise, only the first 4 questions as they appear in your answer book will be marked.
4. All questions are of equal value.
5. Page 5 of the exam is an information sheet.

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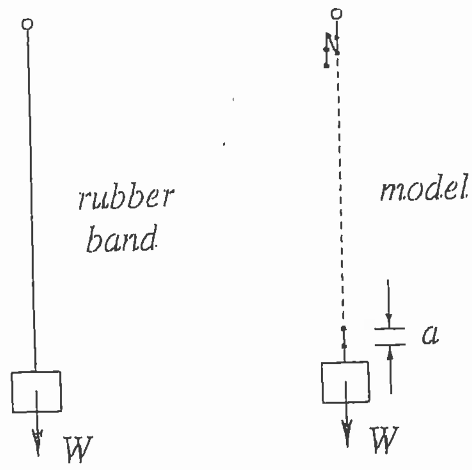
1. A weight W is suspended by a rubber band whose other end is fastened to a peg.



A simple microscopic model of the rubber band is that it consists of a linked polymer chain of N segments joined end to end; each segment has a length a and can be oriented either parallel or antiparallel to the vertical direction. A particular configuration (microstate) of the chain has N_p parallel segments and N_a antiparallel segments, with $N_p + N_a = N$.

[4 marks] (a) Obtain an expression for the energy, E, of the system in terms of N_p and N_a.

[5 marks] (b) How many distinct microstates of the chain are there for a particular energy of the system?

[7 marks] (c) Obtain an expression for the entropy of the chain as a function of its energy.

[9 marks] (d) Use the thermodynamic relation between temperature and entropy to obtain an expression for the mean length $\overline{L}$ of the chain when it is in equilibrium with a heat bath at temperature T.

2. Consider a two-dimensional harmonic solid consisting of N atoms (for example, an ordered array of atoms adsorbed on an inert substrate). The atoms can only vibrate in the plane of the solid. It is found that there is one longitudinal and one transverse sound mode which at long wavelengths have the dispersion relations

$$\omega = s_l k\,, \qquad \omega = s_t k\,,$$

where ω is the angular frequency, k is the wave vector and s_l and s_t are the longitudinal and transverse sound speeds, respectively.

[7 marks] (a) Assuming periodic boundary conditions, what is the density of states for the solid? (Consider the solid to be a square.)

[14 marks] (b) What is the specific heat at constant area of such a solid, and what is its limiting form at low and high temperatures?

[4 marks] (c) How do these results differ from the three-dimensional case?

3. Consider a system of noninteracting particles. The different quantum states accessible to each particle are labelled by the index i and have energy ε_i.

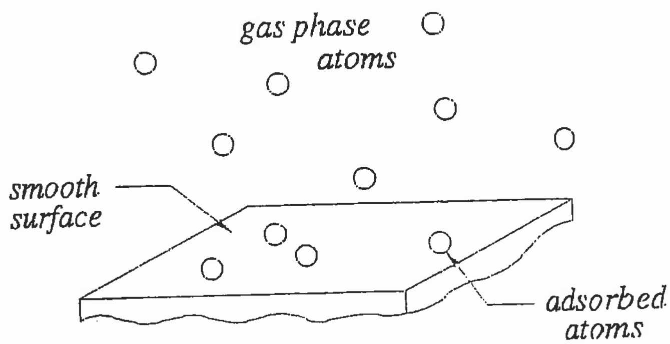
[15 marks] (a) Use the grand canonical ensemble to obtain an expression for the dispersion, $\overline{\Delta n_i^2}$, in the occupation number, n_i, of the i-th state when the particles are fermions and when they are bosons. Express your answer in terms of the mean occupation number $\overline{n}_i$. Comment on the relative width (rms deviation/mean value) of the distribution for the lowest occupied level at *low* temperatures in each case.

[10 marks] (b) Show that the entropy of this system can be expressed as

$$S = -k \sum_i [(1 \mp \overline{n}_i)\ln(1 \mp \overline{n}_i) \pm \overline{n}_i \ln \overline{n}_i]$$

where the upper sign applies to fermions and the lower sign to bosons.

4. Consider a submonolayer film of ^{4}He atoms adsorbed on a plane solid surface of area A. The adsorbed layer is in equilibrium with ^{4}He gas above it.



The surface of the solid is smooth and the density of adsorbed atoms is sufficiently low that the atoms can be thought of as an ideal two-dimensional quantum gas. However, the minimum energy of an atom on the surface is $-\varepsilon_0$ relative to the minimum energy of an atom in the gas phase.

[11 marks] (a) Use the grand canonical ensemble to obtain the equation of state of the gas phase, assuming it to be ideal and classical. In the process, obtain an expression for the chemical potential μ in terms of the pressure, P, and temperature, T, of the gas.

[8 marks] (b) Determine the density of states for the atoms in the adsorbed layer. Assume periodic boundary conditions. Use this result to obtain an expression for the density of atoms on the surface.

[6 marks] (c) Using the fact that the chemical potential of the gas is the same as that of the adsorbed layer, show that the density of adsorbed atoms is given by

$$n_a = \frac{\overline{N}_a}{A} = n_0 \ln\left(\frac{P_0(T)}{P_0(T) - P}\right),$$

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where

$$n_0 = \frac{m}{2\pi\hbar^2\beta}, \qquad P_0(T) = \frac{e^{-\beta\varepsilon_0}}{\beta\lambda^3}; \qquad \lambda = \sqrt{\frac{h^2}{2\pi mkT}}$$

Here, m is the mass of the atoms and $\beta = 1/kT$. Explain why $P_0(T)$ is an upper limit to the pressure with which the adsorbed atoms can be in equilibrium.

5. The molecules in a gas have a permanent electric dipole moment p. When placed in an external electric field $\mathbf{F} = F\hat{\mathbf{z}}$, the energy of a molecule is

$$E = -\mathbf{p}\cdot\mathbf{F} = -pF\cos\theta\,.$$

[12 marks] (a) Show that

$$\overline{p}_x = \overline{p}_y = 0$$

and

$$\overline{p}_z = p\left[\coth\left(\frac{pF}{kT}\right) - \frac{kT}{pF}\right].$$

The bar denotes a canonical ensemble average.

[5 marks] (b) Calculate the electric susceptibility $\chi = \partial\overline{p}_z/\partial F$.

[8 marks] (c) Show that at high temperatures, $\chi \simeq C/T$, and determine C.

6. A photon of momentum $\hbar k$ has an energy $\epsilon_k = c\hbar k$ and can be in either of two possible helicity states. Consider a cavity of volume V containing a gas of photons in equilibrium at the temperature T.

[13 marks] (a) Give a formal expression for the energy of the photon gas and show that the energy takes the form

$$E(V,T) = Vu(T)$$

where $u(T) \propto T^4$.

[12 marks] (b) Using the thermodynamic expression $P = (\partial E/\partial V)_S$, show that the radiation pressure is given by

$$P = \frac{1}{3}u\,.$$

Note that constant entropy, S, means constant occupation probabilities. It may be useful to think of the volume V as a cube of length L on a side.

Information

Thermodynamic Relations

$$S = k \ln \Omega(E) \,, \qquad S = -k \sum_\alpha P_\alpha \ln P_\alpha \,, \qquad \frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N}$$

$$dE = TdS - PdV \,, \qquad F = E - TS \,, \qquad G = F + PV \,, \qquad C_\alpha = T\left(\frac{\partial S}{\partial T}\right)_\alpha$$

Paritition Functions

$$Z_N = \sum_n e^{-\beta E_n(N)}$$

$$Z_N = \frac{1}{N! h^f} \int dp_1 \cdots dp_f \int dq_1 \cdots dq_f e^{-\beta H_N(p,q)}$$

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N \,, \qquad P_\alpha = \frac{e^{-\beta(E_\alpha - \mu N_\alpha)}}{\mathcal{Z}} \,, \qquad \overline{A} = \sum_\alpha A_\alpha P_\alpha$$

$$F = -kT \ln Z_N \,, \qquad P = kT\left(\frac{\partial \ln Z_N}{\partial V}\right)_{T,N} \,, \qquad PV = kT \ln \mathcal{Z}$$

Quantum Distributions

$$f(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} \pm 1} \,, \quad +1 \text{ for fermions, } -1 \text{ for bosons; } \mu = 0 \text{ for photons.}$$

Mathematical Relations

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \,, \qquad \int_0^\infty x^n e^{-\alpha x^2}\, dx = \frac{1}{2}\alpha^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right)$$

$$\Gamma(n+1) = n\Gamma(n) \,, \qquad \Gamma(1) = 1 \,, \qquad \Gamma(1/2) = \sqrt{\pi} \,, \qquad \ln N! \simeq N \ln N - N$$

$$e^x = \sum_{n=0}^\infty \frac{x^n}{n!} \,, \qquad \int e^{\alpha x}\, dx = \frac{e^{\alpha x}}{\alpha} \,, \qquad \int x e^{\alpha x}\, dx = \left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right) e^{\alpha x}$$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \,, \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) \,, \qquad \tanh x = \frac{\sinh x}{\cosh x}$$